22.51 Problem Set 2 (due Wed, Sept. 19)

1 Linear Operators (40 pt)

(**a**). The momentum operator $\hat{\mathbf{p}}$ acts on the spatial part $\psi(\mathbf{x})$ of a quantum state $|\psi\rangle$. For the time being we assume $\psi(\mathbf{x})$ is all there is to $|\psi\rangle$.

$$\hat{\mathbf{p}}|\psi\rangle \;=\; \hat{\mathbf{p}}\psi(\mathbf{x}) \;=\; -i\hbar\nabla\psi(\mathbf{x}). \tag{1}$$

Prove the fundamental relation,

$$[\hat{x}_i, \hat{p}_j] \;=\; i\hbar\delta_{ij}. \tag{2}$$

(**b**). Is $\hat{x}_i$ a constant operator? Why or why not?

(**c**). Prove Jacobi's identity,

$$[\hat{A}, [\hat{B}, \hat{C}]] \;+\; [\hat{B}, [\hat{C}, \hat{A}]] \;+\; [\hat{C}, [\hat{A}, \hat{B}]] \;=\; 0. \tag{3}$$

(**d**). Let $\hat{L}_i \equiv \epsilon_{ijk}\hat{x}_j\hat{p}_k$, or more explicitly,

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \;\; \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \;\; \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \tag{4}$$

prove $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, and write down the other two permutations.

(**e**). Define $\hat{L^2} \equiv \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, prove that $[\hat{L^2}, \hat{L}_x] = [\hat{L^2}, \hat{L}_y] = [\hat{L^2}, \hat{L}_z] = 0$.

(**f**). Let $\hat{p}^2 \equiv \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$, prove that $[\hat{L}_x, \hat{p}^2]$=0.

(**g**). Prove that $[\hat{L}_x, V(r)] = 0$ where $V(r)$ is any central potential.

(**h**). Explain why for a single particle in a central potential, the measurement average $\langle\psi|\hat{\mathbf{L}}|\psi\rangle$ is a constant vector.

Answer:

(**a**). Define $\hat{A} \equiv [\hat{x}_i, \hat{p}_j]$. $\hat{A}$ operating on any state $|\psi\rangle = \psi(\mathbf{x})$ is,

$$\hat{A}|\psi\rangle = [\hat{x}_i, \hat{p}_j]\psi(\mathbf{x}) = x_i(-i\hbar\partial_j\psi(\mathbf{x})) + i\hbar\partial_j\left(x_i\psi(\mathbf{x})\right) = i\hbar(\partial_j x_i)\psi(\mathbf{x}) = i\hbar\delta_{ij}\psi(\mathbf{x}). \tag{5}$$

Therefore $\hat{A} = [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.

(**b**). No. For example, $x^{1/3}$ is an entirely different function from $x^{-2/3}$.

(**c**).

$$[\hat{A},[\hat{B},\hat{C}]] = [\hat{A},\hat{B}\hat{C}-\hat{C}\hat{B}] = \hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}-\hat{B}\hat{C}\hat{A}+\hat{C}\hat{B}\hat{A}, \tag{6}$$

$$[\hat{B},[\hat{C},\hat{A}]] = [\hat{B},\hat{C}\hat{A}-\hat{A}\hat{C}] = \hat{B}\hat{C}\hat{A}-\hat{B}\hat{A}\hat{C}-\hat{C}\hat{A}\hat{B}+\hat{A}\hat{C}\hat{B}, \tag{7}$$

$$[\hat{C},[\hat{A},\hat{B}]] = [\hat{C},\hat{A}\hat{B}-\hat{B}\hat{A}] = \hat{C}\hat{A}\hat{B}-\hat{C}\hat{B}\hat{A}-\hat{A}\hat{B}\hat{C}+\hat{B}\hat{A}\hat{C}. \tag{8}$$

We can inspect that all terms cancel.

(**d**). Two fundamental properties of the Levi-Cevita symbol are,

$$\epsilon_{ijk} = -\epsilon_{ikj}, \quad \epsilon_{ijk}\epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}, \tag{9}$$

where repeated indices in a product (here i) are meant to be summed over. Thus,

$$\begin{aligned}
& [\hat{L}_i, \hat{L}_{i'}] \\
= \;& [\epsilon_{ijk}\hat{x}_j\hat{p}_k, \epsilon_{i'j'k'}\hat{x}_{j'}\hat{p}_{k'}] \\
= \;& \epsilon_{ijk}\epsilon_{i'j'k'}[\hat{x}_j\hat{p}_k, \hat{x}_{j'}\hat{p}_{k'}] \\
= \;& \epsilon_{ijk}\epsilon_{i'j'k'}\left([\hat{x}_j\hat{p}_k, \hat{x}_{j'}]\hat{p}_{k'} + \hat{x}_{j'}[\hat{x}_j\hat{p}_k, \hat{p}_{k'}]\right) \\
= \;& \epsilon_{ijk}\epsilon_{i'j'k'}\left(\hat{x}_j[\hat{p}_k, \hat{x}_{j'}]\hat{p}_{k'} + [\hat{x}_j, \hat{x}_{j'}]\hat{p}_k\hat{p}_{k'} + \hat{x}_{j'}[\hat{x}_j, \hat{p}_{k'}]\hat{p}_k + \hat{x}_{j'}\hat{x}_j[\hat{p}_k, \hat{p}_{k'}]\right) \\
= \;& \epsilon_{ijk}\epsilon_{i'j'k'}\left(-i\hbar\hat{x}_j\delta_{kj'}\hat{p}_{k'} + 0 + i\hbar\hat{x}_{j'}\delta_{jk'}\hat{p}_k + 0\right) \\
= \;& i\hbar\left(-\epsilon_{ijk}\epsilon_{i'kk'}\hat{x}_j\hat{p}_{k'} + \epsilon_{ijk}\epsilon_{i'j'j}\hat{x}_{j'}\hat{p}_k\right) \\
= \;& i\hbar\left((\delta_{ii'}\delta_{jk'} - \delta_{ik'}\delta_{ji'})\hat{x}_j\hat{p}_{k'} - (\delta_{ii'}\delta_{kj'} - \delta_{ij'}\delta_{ki'})\hat{x}_{j'}\hat{p}_k\right) \\
= \;& i\hbar\left(\delta_{ii'}\hat{x}_j\hat{p}_j - \hat{x}_{i'}\hat{p}_i - \delta_{ii'}\hat{x}_k\hat{p}_k + \hat{x}_i\hat{p}_{i'}\right) \\
= \;& i\hbar\left(\hat{x}_i\hat{p}_{i'} - \hat{x}_{i'}\hat{p}_i\right) \\
= \;& i\hbar\epsilon_{ii'k}\epsilon_{kjj'}\hat{x}_j\hat{p}_{j'} \\
= \;& i\hbar\epsilon_{ii'k}\hat{L}_k.
\end{aligned} \tag{10}$$

Thus, when $i = 1$, $i' = 2$, only ϵ_{123} is nonzero, and we obtain,

$$[\hat{L}_x, \hat{L}_y] = i\hbar L_z. \tag{11}$$

Similarly, the permutations

$$[\hat{L}_y, \hat{L}_z] = i\hbar L_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar L_y. \tag{12}$$

(**e**).

$$
\begin{aligned}
& [\hat{L}^2, \hat{L}_i] \\
= \quad & [\hat{L}_j \hat{L}_j, \hat{L}_i] \\
= \quad & \hat{L}_j [\hat{L}_j, \hat{L}_i] + [\hat{L}_j, \hat{L}_i] \hat{L}_j \\
= \quad & \hat{L}_j i\hbar\epsilon_{jik} \hat{L}_k + i\hbar\epsilon_{jik} \hat{L}_k \hat{L}_j \\
= \quad & (i\hbar\epsilon_{jik})(\hat{L}_j \hat{L}_k + \hat{L}_k \hat{L}_j). \qquad (13)
\end{aligned}
$$

The first term is antisymmetric with respect to $j \leftrightarrow k$ permutation, whereas the second term is symmetric with respect to $j \leftrightarrow k$. Because j, k are summed over, the result is 0.

(**f**). First of all,

$$
\begin{aligned}
& [\hat{L}_i, \hat{p}_{i'}] \\
= \quad & [\epsilon_{ijk} \hat{x}_j \hat{p}_k, \hat{p}_{i'}] \\
= \quad & \epsilon_{ijk} [\hat{x}_j, \hat{p}_{i'}] \hat{p}_k \\
= \quad & \epsilon_{ijk} i\hbar \delta_{ji'} \hat{p}_k \\
= \quad & i\hbar\epsilon_{ii'k} \hat{p}_k. \qquad (14)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& [\hat{L}_i, \hat{p}^2] \\
= \quad & [\hat{L}_i, \hat{p}_{i'} \hat{p}_{i'}] \\
= \quad & [\hat{L}_i, \hat{p}_{i'}] \hat{p}_{i'} + \hat{p}_{i'} [\hat{L}_i, \hat{p}_{i'}] \\
= \quad & i\hbar\epsilon_{ii'k} \hat{p}_k \hat{p}_{i'} + \hat{p}_{i'} i\hbar\epsilon_{ii'k} \hat{p}_k \\
= \quad & (i\hbar\epsilon_{ii'k})(\hat{p}_k \hat{p}_{i'} + \hat{p}_{i'} \hat{p}_k), \qquad (15)
\end{aligned}
$$

and it vanishes for the same reason as in (**e**).

(**g**). Similar to (**f**), we can prove that,

$$
[\hat{L}_i, \hat{x}_{i'}] = i\hbar\epsilon_{ii'k} \hat{x}_k, \quad [\hat{L}_i, \hat{x}^2] = 0. \qquad (16)
$$

If V is a function of r, then it must also be a function of r^2. Let,

$$
V(r) \equiv W(r^2). \qquad (17)
$$

Because $\hat{L}_i$ contains only one $\hat{p}_\alpha$, it is easy to show that,

$$[\hat{L}_i, W(r^2)] \;=\; W'(r^2)[\hat{L}_i, r^2] \;=\; W'(r^2)[\hat{L}_i, \hat{x}^2] = 0, \tag{18}$$

so indeed,

$$[\hat{L}_i, V(r)] \;=\; 0. \tag{19}$$

(**h**). The Hamiltonian operator is,

$$\hat{\mathcal{H}} \;=\; \hat{T} + \hat{V} \;=\; \frac{\hat{p}^2}{2m} + V(r). \tag{20}$$

Because $\hat{L}_i$ commutes with both $\hat{T}$ and $\hat{V}$, it commutes with $\hat{\mathcal{H}}$. Furthermore, the definition of $\hat{L}_i$ in terms of elementary operators $\{\hat{x}_\alpha\}$ and $\{\hat{p}_\beta\}$: $\hat{L}_i \equiv \epsilon_{ijk}\hat{x}_j\hat{p}_k$, contains no explicit dependence on t (that is, ϵ_{ijk} is a constant factor), so,

$$\frac{\partial \hat{L}_i}{\partial t} \;=\; \left(\frac{\partial \epsilon_{ijk}}{\partial t}\right)\hat{x}_j\hat{p}_k \;=\; 0. \tag{21}$$

Therefore, in the Heisenberg picture,

$$\frac{d\hat{L}_i}{dt} \;=\; \frac{1}{i\hbar}[\hat{L}_i, \hat{\mathcal{H}}] \;+\; \frac{\partial \hat{L}_i}{\partial t} \;=\; 0 \;+\; 0 \;=\; 0. \tag{22}$$

And $\langle \hat{L}_i \rangle$ is time independent.

2 Operator Functions (20 pt)

Question:

(**a**). Operator $\hat{A}$ has eigenvalues $\{\lambda_i\}$. Let $\hat{B} \equiv f(\hat{A})$. Prove that $\hat{B}$ has eigenvalues $\{f(\lambda_i)\}$.

(**b**). If operator $\hat{U}$ satisfies,

$$\hat{U}\hat{U}^+ \;=\; \hat{U}^+\hat{U} \;=\; \hat{I}, \tag{23}$$

it is called a unitary operator. Suppose $\hat{A}$ is Hermitian. Prove $\exp(i\hat{A}t)$ $(t \in \mathbf{R})$ is unitary.

Answer:

(**a**). Suppose,

$$\hat{A}|\lambda_i\rangle \;=\; \lambda_i|\lambda_i\rangle, \tag{24}$$

and by definition,

$$f(\hat{A}) \quad \equiv \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(x=0)}{n!} \hat{A}^n. \tag{25}$$

Then,

$$\hat{B}|\lambda_i\rangle = \sum_{n=0}^{\infty} \frac{f^{(n)}(x=0)}{n!} \hat{A}^n |\lambda_i\rangle = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \lambda_i{}^n |\lambda_i\rangle = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \lambda_i{}^n \right) |\lambda_i\rangle = f(\lambda_i)|\lambda_i\rangle.$$

So the eigenvalues of $\hat{B}$ are $\{f(\lambda_i)\}$.

(**b**). Define,

$$\hat{U} \quad \equiv \quad \exp(i\hat{A}t) \quad = \quad \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \hat{A}^n. \tag{26}$$

When we take the Hermitian conjugate,

$$\hat{U}^+ \quad = \quad \sum_{n=0}^{\infty} \frac{(it)^{*n}}{n!} \hat{A}^{+n} \quad = \quad \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \hat{A}^n. \tag{27}$$

The latter is because $\hat{A}$ is Hermitian: $\hat{A}^+ = \hat{A}$. So there is,

$$\hat{U}\hat{U}^+ \quad = \quad \left(\sum_{n=0}^{\infty} \frac{(it)^n}{n!} \hat{A}^n \right) \left(\sum_{m=0}^{\infty} \frac{(-it)^m}{m!} \hat{A}^m \right) \quad = \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(it)^n (-it)^m}{n!m!} \hat{A}^{n+m}. \tag{28}$$

Define $l = n + m$, and we collect various terms of equal l, as,

$$\hat{U}\hat{U}^+ \quad = \quad \sum_{l=0}^{\infty} \frac{c_l}{l!} \hat{A}^l, \quad c_l \quad \equiv \quad \sum_{n=0}^{l} \frac{l!(it)^n (-it)^{l-n}}{n!(l-n)!}. \tag{29}$$

So c_l is in fact the lth binomial expansion coefficient of $(it - it)^l$, and would be zero unless $l = 0$, and in which case it would be 1,

$$c_l \quad = \quad \delta_{l0}. \tag{30}$$

So,

$$\hat{U}\hat{U}^+ \quad = \quad \frac{c_0}{0!} \hat{A}^0 \quad = \quad \hat{I}, \tag{31}$$

and $\hat{U}$ is unitary for arbitrary Hermitian $\hat{A}$.

3 Quantum Fluctuation (20 pt)

Question: Using Quantum Postulates 1 and 2, *prove* that the measurement variance of an observable A is given by,

$$\sigma^2(A) \; = \; \langle\psi|(\hat{A} - \bar{A})^2|\psi\rangle, \tag{32}$$

where $|\psi\rangle$ is the current state, and $\bar{A} \equiv \langle\psi|\hat{A}|\psi\rangle$ is the measurement average.

Answer: The Quantum Postulates 1 and 2 are *much more* than simply stating that $\bar{A} = \langle\psi|\hat{A}|\psi\rangle$. They in fact fully specify the *distribution* of measurement outcome. In the case of $\hat{A}$ having a discrete eigenvalue spectrum $\{a_n\}$, the probability of getting a particular measurement outcome a_n is asserted to be $P_n = |\langle a_n|\psi\rangle|^2$. We have shown in class that,

$$\sum_n P_n \; = \; 1, \quad \bar{A} \; = \; \sum_n a_n P_n \; = \; \langle\psi|\hat{A}|\psi\rangle. \tag{33}$$

The operational definition of measurement outcome variance is,

$$\sigma^2(A) \; \equiv \; \sum_n (a_n - \bar{A})^2 P_n. \tag{34}$$

But it can be simplified as,

$$\begin{aligned}
\sigma^2(A) \quad &= \quad \sum_n (a_n - \bar{A})^2 |\langle a_n|\psi\rangle|^2 \\
&= \quad \sum_n (a_n - \bar{A})^2 \langle\psi|a_n\rangle\langle a_n|\psi\rangle \\
&= \quad \sum_n \langle\psi|(a_n - \bar{A})^2|a_n\rangle\langle a_n|\psi\rangle \\
&= \quad \sum_n \langle\psi|(\hat{A} - \bar{A})^2|a_n\rangle\langle a_n|\psi\rangle \\
&= \quad \langle\psi|(\hat{A} - \bar{A})^2| \left(\sum_n |a_n\rangle\langle a_n|\right) |\psi\rangle \\
&= \quad \langle\psi|(\hat{A} - \bar{A})^2 \hat{I}|\psi\rangle \\
&= \quad \langle\psi|(\hat{A} - \bar{A})^2|\psi\rangle.
\end{aligned} \tag{35}$$

The proof in the case of $\hat{A}$ having a continuous eigenvalue spectrum is similar and is left to the reader.

4 Heisenberg Uncertainty Principle (20 pt)

Question: Prove that,

$$\sigma^2(A)\sigma^2(B) \;\; \geq \;\; \frac{1}{4}\langle\psi|[\hat{A},\hat{B}]|\psi\rangle^2 \tag{36}$$

Answer: When $\hat{A}$ and $\hat{B}$ are Hermitian, let us define,

$$\hat{C} \;\; \equiv \;\; \lambda\hat{A} + i\hat{B}, \quad \lambda \in \mathbf{R}, \tag{37}$$

Then,

$$\hat{C}^{+} \;\; \equiv \;\; \lambda\hat{A} - i\hat{B}, \tag{38}$$

and,

$$\hat{C}^{+}\hat{C} \;\; = \;\; \lambda^2\hat{A}^2 + i\lambda\hat{A}\hat{B} - i\lambda\hat{B}\hat{A} + \hat{B}^2. \tag{39}$$

For any $|\psi\rangle$, define $|\psi'\rangle \equiv \hat{C}|\psi\rangle$, there is,

$$0 \;\; \leq \;\; \langle\psi'|\psi'\rangle \;\; = \;\; \langle\psi|\hat{C}^{+}\hat{C}|\psi\rangle. \tag{40}$$

When we expand out $\hat{C}^{+}\hat{C}$ using (39), there is,

$$0 \;\; \leq \;\; \lambda^2\langle\hat{A}^2\rangle + i\lambda\langle[\hat{A},\hat{B}]\rangle + \langle\hat{B}^2\rangle, \tag{41}$$

which must hold true for any $\lambda \in \mathbf{R}$. From elementary algebra we know this can only be possible if,

$$\langle\hat{A}^2\rangle\langle\hat{B}^2\rangle \;\; \geq \;\; \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2. \tag{42}$$

At this moment it is still not in the form (36) that we want. However, given any $\hat{A}$, $\hat{B}$, we can define new operators,

$$\hat{\mathcal{A}} \;\equiv\; \hat{A} - \bar{A}, \quad \hat{\mathcal{B}} \;\equiv\; \hat{B} - \bar{B}. \tag{43}$$

Since $\bar{A}$,$\bar{B}$ are real, $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$ are still Hermitian. Furthermore, there is,

$$[\hat{\mathcal{A}},\hat{\mathcal{B}}] \;\; = \;\; [A,B]. \tag{44}$$

Plugging $\hat{\mathcal{A}}$,$\hat{\mathcal{B}}$ into (42), we will arrive at (36).

5 False Question (15 pt)

Quantum Postulate 1 says that any measurement influences the state. However, suppose there is a measurement A, but *no one knows the result*, then what happens?

In probability theory there is a difference between *a priori* and *a posteriori* probability. Show that if A is measured first but no one knows the outcome, then an ensuing measurement B would be *no different* from the case where A is not measured at all.

However, if we know the outcome of the A measurement is a_n - one of $\hat{A}$'s many eigenvalues, then everything will be different, right?

As for the joint probability of getting a certain (a_n, b_m) pair, does it make a difference between A measured first, B second, and the converse?

Wrong Answer: Whenever a measurement is performed, the quantum state would instantaneously change from the previous $|\psi\rangle$ to one of $\hat{A}$'s eigenstate, $|a_n\rangle$, depending on which a_n the experimentalist sees (the probability of getting a particular a_n is $P_n = |\langle a_n|\psi\rangle|^2$). However, if the experimentalist was not able to see the outcome, then the current quantum state remains unknown to him. What he can claim though, is that the *probability* that the current quantum state being in $|a_n\rangle$ is P_n.

Therefore, if he continues to make the next measurement B, there is P_n probability that the state is $|a_n\rangle$, and then the probability of getting a b_m result is $|\langle b_m|a_n\rangle|^2$. Therefore, the total probability that a b_m result is obtained is,

$$p = \sum_n P_n |\langle b_m|a_n\rangle|^2 = \sum_n \langle b_m|a_n\rangle P_n \langle a_n|b_m\rangle. \tag{45}$$

However,

$$\sum_n |a_n\rangle P_n \langle a_n| = \sum_n |a_n\rangle\langle a_n|\psi\rangle\langle\psi|a_n\rangle\langle a_n| = |\psi\rangle\langle\psi|. \tag{46}$$

Thus,

$$p = \sum_n \langle b_m|\psi\rangle\langle\psi|b_m\rangle = |\langle\psi|b_m\rangle|^2, \tag{47}$$

as if A was not measured at all.

If the experimentalist knows it's a_n, then of course the distribution of B would be $|\langle a_n|b_m\rangle|^2$, which is vastly different from $|\langle\psi|b_m\rangle|^2$.

If $\hat{A}$ and $\hat{B}$ do not commute, then it makes a difference. The first joint probability is $|\langle\psi|a_n\rangle|^2|\langle a_n|b_m\rangle|^2$, the second is $|\langle\psi|b_m\rangle|^2|\langle b_m|a_n\rangle|^2$.